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APPROXIMATE ANALYTICAL METHOD OF CALCULATING THE TRAJECTORIES OF AN AIRCRAFT IN THE ATMOSPHERE*

M.N. IVANOV, A.I. MARTYNOV and N.L. SOKOLOV

An approximate method of calculating the trajectories of an aircraft in the atmosphere is proposed which takes into account the ablation of the aircraft mass and rotation of the planet. The effectiveness of the method is shown in the calculation of aircraft in the high-density atmosphere at high velocities. The results are illustrated by a number of numerical examples.

1. Let us consider the system of differential equations that define the motion of an aircraft in the atmosphere /1,2/

$$\frac{dV}{dt} = -\frac{C_x S\rho V^2}{2m} + (\omega^2 r - g)\sin\theta \tag{1.1}$$

$$\frac{d\theta}{dt} = \frac{C_y S\rho V}{2m} + \left(\frac{\omega^2 r}{V} + \frac{V}{r} - \frac{g}{V}\right)\cos\theta + 2\omega$$

$$\frac{dh}{dt} = V\sin\theta, \quad \frac{dL}{dt} = V\cos\theta\frac{R}{r}, \quad \frac{dm}{dt} = -\frac{C_1 S_1 q_\Sigma}{\eta}$$

where V is the aircraft flight velocity, θ is the velocity vector angle of inclination to the local horizon, h is the flight altitude, r is the distance of the aircraft from the planet center, m is the aircraft mass, ρ is the atmosphere density, C_x and C_y are, respectively, the drag and lift, S is the aircraft median cross sectional area, ω is the angular velocity of the planet rotation, g is the acceleration force of gravity, t is the time of flight, q_Σ is the total heat flux at the vehicle critical point, η is the effective enthalpy, C_1 is the averaged coefficient of nonuniformity of the ablated mass from the aircraft surface, S_1 is the area of ablated surface of the aircraft, and L is the flight range.

System (1.1) defines the plane motion of the aircraft in equatorial entry, with the planet rotation and the ablation of the mass of heat insulation coating from the vehicle surface taken into account.

A solution of system (1.1) can be obtained with the necessary accuracy by numerical methods of integration. This is, however, linked with large expenditure of computer time, particularly in the case of parametric calculations or in solving boundary value problems, when a large number of iterations is required /1–3/. Hence, in many instances the necessity arises to have a solution of the system in analytic form. Simpler analytic form of differential equations (without taking into acoount the planet rotation and the ablation of thermal insulation from the vehicle coating) approximate analytical methods were applied in /4–15/. It should be noted that the use of these methods for analyzing the motions of aircraft in atmospheres of high density (atmospheres of Venus and Jupiter) or at high entry velocities, lead to large relative errors (for problems of descent of a vehicle in the atmosphere of Jupiter the calculated data may differ from numerically obtained data by more than 20%).

2. The analysis of aircraft trajectories in the atmosphere of the Earth and other planets allows the introduction of the following assumptions:

$$\left|\frac{C_x S\rho V^2}{2m}\right| \gg |g\sin\theta| + |\omega^2 r\sin\theta|, \quad R \gg h$$

$$\theta \ll 1, \ \rho(h) = \rho_0 \exp(-\beta h), \ q_\Sigma = {}^1/_2 \alpha\rho V^3$$

where α is the coefficient that defines the absorbing capacity of the heat insulating coating.

Omitting the time t and taking into account the indicated assumptions we write system (1.1) in the form

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$$\frac{d\theta}{dV}=-\frac{K}{V}-\frac{2mM}{C_xSV},\quad \frac{d\rho}{dV}=\frac{2m\theta\rho}{C_xSV} \tag{2.1}$$
$$\frac{dm}{dV}=\frac{C_1S_1\alpha Vm}{\eta C_xS},\quad \frac{dL}{dt}=-\frac{2m}{C_xS\rho V}$$
$$M=\left(1-\frac{gr}{V^2}+\frac{\omega^2r^2}{V^2}+\frac{2\omega r}{V}\right)\frac{1}{\rho r},\quad K=\frac{C_y}{C_x}$$

The motion time is determined by the relation

$$t=-\frac{2}{C_xS}\int\limits_{V_0}^{V}\frac{m(V)}{\rho(V)V^2}\,dV$$

We further assume function M to be piecewise-constant in finite intervals of variation of argument V.

Integrating the third of equations of system (2.1), we determine the dependence of the aircraft mass on motion velocity V as

$$m(V)=A\exp(BV)^2 \tag{2.2}$$
$$A=m_0\exp\left(-\frac{\varkappa V_0^2}{2C_xS}\right),\quad B=\frac{\varkappa}{2C_xS},\quad \varkappa=\frac{C_1S_1\alpha}{\eta}$$

The dependence on the angle of inclination θ of the velocity vector to the local horizon is obtained from the first of equations of system (2.1)

$$\theta(V)=\theta_0-K\ln\frac{V}{V_0}-\frac{2MA}{C_xS}I_1,\quad I_1=\int\limits_{V_0}^{V}\frac{\exp(BV^2)}{V}\,dV \tag{2.3}$$

From the second of Eqs.(2.1), with allowance for (2.2) and (2.3), we obtain the relation between the vehicle velocity V and the atmospheric density ρ that corresponds to the flight altitude of aircraft in the form

$$\rho(V)=\rho_0+\frac{2\beta A}{C_xS}[(\theta_0+K\ln V_0)I_1-KI_2-2MAI_3] \tag{2.4}$$
$$I_2=\int\limits_{V_0}^{V}\frac{\exp(BV^2)\ln V}{V}\,dV,\quad I_3=\int\limits_{V_0}^{V}\frac{\exp(BV^2)I_1}{V}\,dV.$$

Note that relation (2.4) actually determines the flight velocity variation with altitude.

Formulas (2.2), (2.3), and (2.4) enable us to determine the aircraft trajectory in the atmosphere, when the initial conditions, the characteristics of the aircraft, and parameters of the atmosphere are known. For determining the trajectories of a ballistic type vehicle it is necessary to set $K=0$ in formulas (2.2), (2.3) and (2.4).

3. The calculation of aircraft trajectories was carried out using formulas (2.2), (2.3), and (2.4) by expanding the exponential terms of integrands in series and rejecting the higher terms of the latter. Let us analyze the effect of number l of retained terms of the series on the accuracy of calculations.

Investigations carried out had shown that when $l=2$ in numerous calculations of aircraft trajectories in atmospheres of several planets the error of computation did not exceed 10%, as compared with similar data obtained by numerical methods (see, e.g., /3/). In particular, the computation of vehicle motion in the Earth atmosphere shows a relative error not exceeding 7%, while for Jupiter it is 10%. The time taken for numerical computations using the proposed approximate analytic formulas is reduced by 15–20 times of that required for numerical calculations of solution of the input problem. When $l=3$, the error does not exceed 5%, at $l=5$ it does not exceed 4%, and when $l=2$ it is 2.5%.

A comparative analysis of the calculation exactness of trajectory parameters carried out by the approximate method (for $l=2$) with other known analytic solutions /4–14/ show that for the motion of any planet in the atmosphere of the Solar system the use of the proposed analytical method yields the most exact data (even for $l=2$). The errors of determination of final values of the trajectory angle θ_K and altitude h_K, the maximum values of overload (n_{max}), temperature (T_{max}), and the total heat flux ($q_{\Sigma max}$), calculated using the methods proposed in /4–14/ are fairly large, being equal to 15-20% (for Jupiter). Determination of the same parameters using the proposed here methods, the errors do not exceed 10%.

At the stage of preliminary calculations of aircraft trajectories in the atmospheres of planets it is, thus, possible to restrict oneself to the value of $l=2$. In that case the calculation formulas (2.2), (2.3), and (2.4) for the aircraft flight with constant lift/drag ratio aerodynamic quality, assume to form

$$m(V) = A\exp(BV^2), \quad \theta(V) = \theta_0 - K\ln\frac{V}{V_0} - \frac{2MA}{C_xS}G_1(V) \tag{3.1}$$

$$\rho(V) = \rho_0 + \frac{2\beta A}{C_xS}\{G_1(V)[\theta_0 + K\ln V_0 - MAG_1(V)] - KG_2(V)\}$$

$$G_1(V) = \ln\frac{V}{V_0} + \frac{B}{8}(V^2 - V_0^2)(4 + BV^2 + BV_0^2)$$

$$G_2(V) = \frac{1}{2}\Big(\ln^2 V - \ln^2 V_0 + BV^2\ln V - BV_0^2\ln V_0 - \frac{BV^2}{2} + \frac{BV_0^2}{2} + \frac{B^2V^4\ln V}{4} - \frac{B^2V_0^4\ln V_0}{4} - \frac{B^2V^4}{16} + \frac{B^2V_0^4}{16}\Big)$$

For ballistic trajectories the second and third of these equations assume the form

$$\theta(V) = \theta_0 - \frac{2MA}{C_xS}G_1(V)$$

$$\rho(V) = \rho_0 + \frac{2\beta A}{C_xS}G_1(V)[\theta_0 - MAG_1(V)]$$

When determining the parameters of aircraft trajectories in the atmosphere of planets, where the ablation of the thermal protection coating is insignificant (e.g., some cases of motion in the Mars atmosphere) the calculation formulas (3.1) reduce to the simpler form

$$\theta(V) = \theta_0 - \left(K + \frac{2mM}{C_xS}\right)\ln\frac{V}{V_0} \tag{3.2}$$

$$\rho(V) = \rho_0 + \frac{2m\beta}{C_xS}\ln\frac{V}{V_0}\left[\theta_0 - 2\left(K + \frac{2mM}{C_xS}\right)\ln\frac{V}{V_0}\right].$$

Let us analyze the exactness of calculation of motion parameters in conformity with formulas (3.2), and compare these with the analytical methods of /4– 14/. The errors of determination of motion parameters using the proposed here method do not exceed 10%, which is lower than the respective errors obtained by using methods of /4– 6/ and /11– 14/, and somewhat larger than those calculated by the methods of /7– 10/.

Thus, for example, the errors of calculation of final values of parameters θ_K and V_K of an aircraft descent trajectories in the atmosphere of Mars does not exceed 7% when using proposed approximate analytical method, and does not exceed 4% in the case of methods of /7– 10/ and 16% in that of /4– 6,11– 14/.

4. The use of the approximate analytical method of calculating aircraft trajectories in the atmosphere are most effective when carrying out mass parametric investigations and in solving multiple iteration boundary value problems occurring in optimization of a given functional. As previously noted, the proposed method can be used for cases when the vehicle moves with constant or zero value of the lift/drag coefficient. This, however, does not preclude the possibility of applying it to solving variational problems. For instance, as shown in /1, 2/, solution of the problems of optimal control of a vehicle of the gliding type reduces to the search of instants of switching the control parameter, the left/drag ratio from one extremal value to another. This theoretically permits to carry out consecutively the calculations for those sections of flight, where it takes place with constant lift/drag ratio, and then join such sections. In the presence of constraints on the trajectory parameters, or a two-channel control by the angles of bank and attack, the proposed analytical method can be used in the search of first approximation.

Below, are presented the solutions of a few known problems of optimal control of aircraft motions in the Jupiter atmosphere, viz. minimization of the maximum value of velocity overload n_{max}, of the over-all heat flux at the critical point Q_k, maximization of the finite descent altitude of the vehicle h_K with constraints on the maximum permissible overload ($n \leqslant n^*$), and the maximization of departure velocity of the vehicle from the atmosphere, when injecting into a specified orbit of the artifical satellite of Jupiter. The following input conditions were assumed: $V_0 = 6\cdot10^4$ m/s $\theta_0 = -0.14$ rad $P_x = 200$ kg/m^2 $K = 0.5$, $\rho_0 = 0.152$ kg/m^3 $\beta = 4.6\cdot10^{-5}$ m^{-1}. Parameters $V_0, \theta_0, P_x, K, \rho_0, \beta$ were varied in the following range: $5\cdot10^4$ m/s $\leqslant V_0 \leqslant 7\cdot10^4$ m/s 0.07 rad $\leqslant |\theta_0| \leqslant 0.21$ rad $0.2 \leqslant K \leqslant 0.6$, 100 kg/m^2 $\leqslant P_x \leqslant 400$ kg/m^2 0.124 kg/m^3 $\leqslant \rho_0 \leqslant 0.24$ kg/m^3 $3.28\cdot10^{-5}$ m^{-1} $\leqslant \beta \leqslant 5.5\cdot10^{-5}$ m^{-1}.

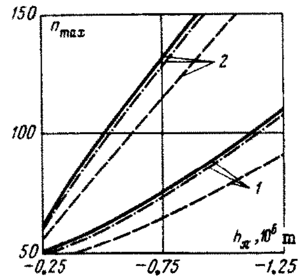

Fig.1

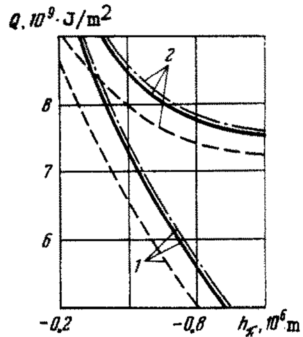

Fig.2

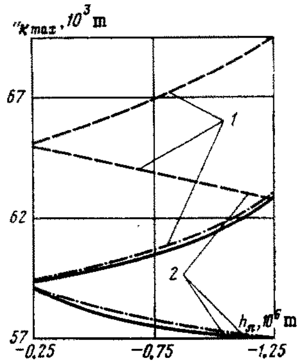

Fig.3

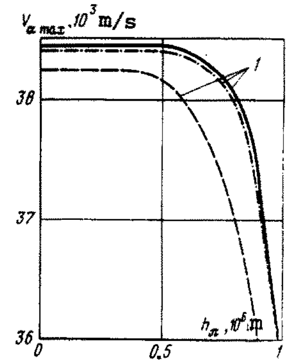

Fig.4

The results in Figs.1– 4 show that qualitatively numerical (solid lines) and approximate (dash lines) solutions coincide, while quantitatively the approximate calculations yield an error of order 10% (curves 1 relate to the optimal control, and 2 to zero-lift trajectories).

5. The comparative analysis of results of calculated trajectories with the use of the proposed approximate analytical method with the numerical solution revealed that, depending on the concrete problem, the approximate data at the trajectory final point are always either greater or smaller of the respective numerical results. This feature enables us to introduce in calculation formulas correction polynomials that compensate for systematic errors and increase the accuracy of solution of the problem. For this it is necessary to analyze the error (Λ) occurring in the approximate solution, and find the dependence of errors on conditions at entry (V_0, θ_0), on the aircraft parameters (K, P_x), and of the characteristics of the atmosphere (ρ_0, β).

Generally, the error Λ can be represented in the form of a polynomial of all varied parameters

$$\Lambda = \sum_{i_j,\, j=1,2,\ldots,6}^{n_j} A_{i1}A_{i2}A_{i3}A_{i4}A_{i5}A_{i6} V_{0i_1}^{i_1}\theta_{0i_2}^{i_2}P_{xi_3}^{i_3}K_{i_4}^{i_4}\rho_{0i_5}^{i_5}\beta_{i_6}^{i_6}$$

Investigations of a wide class of problems of aircraft motion in the atmosphere of planets have shown that it is sufficient to introduce in computation formulas second order correction polynomials, which allows to increase the accuracy of calculation of trajectories to 3% (see, e.g., the dash-dot lines in Figs.1– 4).

Note that the correction polynomials are calculated for each specific class of problems. Obviously it is most expedient to introduce these in computation formulas in investigations of large set of trajectories, where a high accuracy of computations is required (e.g., in solving multi-iterational boundary value problems, in parametric investigations, etc.).

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